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Existence of the solutions for a class of nonlinear fractional order three-point boundary value problems with resonance

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Abstract

A class of nonlinear fractional order differential equation

$$D^{\alpha}_{0+}u(t)+f(t,u(t))=0,\quad 0<t<1,$$
$$u(0)=0,\qquad u(1)=\frac{1}{\eta^{\alpha-1}}u(\eta)$$

is investigated in this paper, where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative of order $1<\alpha\leq 2$, $0<\eta<1$, $f\in C([0,1]\times R,R)$. Using intermediate value theorem, we obtain a sufficient condition for the existence of the solutions for the above fractional order differential equations.

1 Introduction

Consider the following boundary value problem

$$D^{\alpha}_{0+}u(t)+f(t,u(t))=0,\quad 0<t<1, \tag{1.1}$$
$$u(0)=0,\qquad u(1)=\frac{1}{\eta^{\alpha-1}}u(\eta), \tag{1.2}$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative of order $1<\alpha\leq 2$, $0<\eta<1$ and $f\in C([0,1]\times R,R)$.

In the last few decades, many authors have investigated fractional differential equations which have been applied in many fields such as physics, mechanics, chemistry, engineering etc. (see references [1, 6, 10, 21–23]). Especially, many works have been devoted to the study of initial value problems and bounded value problems for fractional order differential equations [12, 13, 15, 24].

Recently, the existence of positive solutions of fractional differential equations has attracted many authors' attention [2–5, 8, 9, 12, 14, 17–20, 25, 26]. Using some fixed point theorems, they obtained some nice existence conditions for positive solutions.

In more recent works, Jiang and Yuan [7] considered the following boundary value problem of fractional differential equations

$$D^{\alpha}_{0+}u(t)+f(t,u(t))=0,\quad 0<t<1, \tag{1.3}$$

$$u(0) = u(1) = 0, \tag{1.4}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order $1 < \alpha < 2$ and $f : [0,1] \times R_+ \to R_+$ is continuous. Using some properties of the Green function $G(t,s)$, they obtain some new sufficient conditions for the existence of positive solutions for the above problem.

Further, Li, Luo, and Zhou [4] investigated the following fractional order three-point boundary value problems

$$D_{0+}^{\alpha} u(t) + f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1, \tag{1.5}$$

$$u(0) = 0, \qquad D_{0+}^{\beta} u(1) = a D_{0+}^{\beta} u(\xi), \tag{1.6}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order $1 < \alpha \le 2$, $0 \le \beta \le 1$, $0 \le a \le 1$, $\xi \in (0,1)$, $a\xi^{\alpha-\beta-2} \le 1 - \beta$, $0 \le \alpha - \beta - 1$, and $f : [0,1] \times R_+ \to R_+$ is continuous.

In this paper, we discuss the boundary value problem (1.1)-(1.2). Using some properties of the Green function $G(t,s)$ and intermediate value theorem, we establish some sufficient conditions for the existence of the positive solutions of the problem (1.1)-(1.2).

The paper is arranged as follows: In Section 2, we introduce some definitions for fractional order differential equations and give our main results for the boundary value problem (1.1)-(1.2). We give some lemmas for our results in Section 3. In Section 4, we prove our main result; and finally, we give an example to illustrate our results.

2 Main results

In this section, we introduce some definitions and preliminary facts which are used in this paper.

Definition 2.1 ([1, 10]) The fractional integral of order α with the lower limit t_0 for a function f is defined as

$$I_{t_0+}^{\alpha}\bigl(f(t)\bigr) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{1-\alpha}}\, ds, \quad t > t_0, \alpha > 0,$$

provided that the integral on the right-hand side is point-wise defined on $[t_0, \infty)$, where Γ is the Gamma function.

Definition 2.2 ([1, 10]) Riemann-Liouville derivative of order α with the lower limit t_0 for a function $f : [0,\infty) \to R$ can be written as

$$D_{t_0+}^{\alpha}\bigl(f(t)\bigr) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{\alpha+1-n}}\, ds, \quad t > t_0, n-1 < \alpha \le n,$$

where n is a positive integer.

We call the function $u(t)$ a solution of (1.1)-(1.2) if $u(t) \in C[0,1] \cap L[0,1]$ with a fractional derivative of order α belongs to $C[0,1] \cap L[0,1]$ and satisfies Equation (1.1) and the boundary condition (1.2).

We also need to introduce some lemmas as follows, which will be used in the proof of our main theorems.

Lemma 2.1 ([26]) *Assume that* $h(t) \in C(0,1) \cap L(0,1)$ *with a fractional derivative of order* $\alpha > 0$ *belongs to* $C(0,1) \cap L(0,1)$. *Then, the fractional equation*

$$D^{\alpha}_{t_0+}\big(h(t)\big) = 0 \tag{2.1}$$

has solutions

$$h(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad c_i \in R, i = 1,2,\ldots,n, n = [\alpha] + 1. \tag{2.2}$$

Lemma 2.2 ([26]) *Assume that* $h(t) \in C(0,1) \cap L(0,1)$ *with a fractional derivative of order* $\alpha > 0$ *belongs to* $C(0,1) \cap L(0,1)$. *Then*

$$I^{\alpha}_{t_0+} D^{\alpha}_{t_0+} h(t) = h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n} \tag{2.3}$$

for some $c_i \in R$, $i = 1,2,\ldots,n$, $n = [\alpha] + 1$.

Lemma 2.3 ([16]) *Suppose that* X *be a Banach space,* $C \subset X$ *is closed and convex. Assume that* U *is a relatively open subset of* C *with* $0 \in U$, *and* $T : \overline{U} \to C$ *is a completely continuous operator. Then, either*

(i) T *has a fixed point in* $\overline{U}$, *or*
(ii) there exist $u \in \partial U$ *and* $\gamma \in (0,1)$ *with* $u = \gamma Tu$.

Throughout this paper, we assume that $f(t,u)$ satisfies the following:

(H) $f(t,u) \in C([0,1] \times R, R)$, *and there exist two positive functions* $a(t) \in C([0,1], R_+)$ *and* $b(t) \in C([0,1], R_+)$ *such that*

$$\big|f\big(t, t^{\alpha-1}u\big)\big| \le a(t) + b(t)|u|^p, \quad t \in [0,1], \tag{2.4}$$

where $0 \le p \le 1$. *Furthermore,*

$$\lim_{u \to \pm\infty} f\big(t, t^{\alpha-1}u\big) = \pm\infty \tag{2.5}$$

for any $t \in (0,1)$.

We have our main results:

Theorem 2.1 *Suppose that (H) holds. If*

$$\int_0^1 G^*(s,s) b(s)\, ds < 1, \tag{2.6}$$

then the boundary value problem (1.1)-(1.2) has at least one solution, where

$$G^*(s,s) = \frac{1}{\Gamma(\alpha)(1-\eta^{\alpha-1})} \begin{cases} (1-s)^{\alpha-1} - (\eta - s)^{\alpha-1}, & 0 \le s \le \eta, \\ (1-s)^{\alpha-1}, & \eta \le s \le 1. \end{cases}$$

3 Some lemmas

Let $\Omega = C[0,1]$, $u \in \Omega$ equipped the norm

$$\|u\| = \sup_{0\leq t\leq 1} \bigl|u(t)\bigr|, \tag{3.1}$$

then Ω is a Banach space.

We first give some lemmas as follows:

Lemma 3.1 *Problem (1.1)-(1.2) is equivalent to the following integral equation*

$$u(t) = \int_0^1 G(t,s) f\bigl(s,u(s)\bigr)\,ds + u(1)t^{\alpha-1}, \tag{3.2}$$

where

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-1} - t^{\alpha-1}(\eta-s)^{\alpha-1} - (1-\eta^{\alpha-1})(t-s)^{\alpha-1}}{\Gamma(\alpha)(1-\eta^{\alpha-1})}, \\ \quad 0 \leq s \leq \min\{t,\eta\} \leq 1; \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1} - t^{\alpha-1}(\eta-s)^{\alpha-1}}{\Gamma(\alpha)(1-\eta^{\alpha-1})}, \quad 0 \leq t \leq s \leq \eta \leq 1; \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1} - (1-\eta^{\alpha-1})(t-s)^{\alpha-1}}{\Gamma(\alpha)(1-\eta^{\alpha-1})}, \quad 0 \leq \eta \leq s \leq t \leq 1; \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\eta^{\alpha-1})}, \quad 0 \leq \max\{t,\eta\} \leq s \leq 1. \end{cases} \tag{3.3}$$

Proof The sufficiency is obvious, we only need to prove the necessity.

Suppose that $u(t)$ is a solution of the problem (1.1)-(1.2). Integrating both sides of (1.1) of α order from 0 to t with respect to t, it follows that

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\bigl(s,u(s)\bigr)\,ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}. \tag{3.4}$$

According to (1.2) and (3.4), we have

$$\begin{aligned} &c_2 = 0, \\ &c_1 = \frac{1}{\Gamma(\alpha)(1-\eta^{\alpha-1})} \biggl\{ \int_0^1 (1-s)^{\alpha-1} f\bigl(s,u(s)\bigr)\,ds - \int_0^\eta (\eta-s)^{\alpha-1} f\bigl(s,u(s)\bigr)\,ds \biggr\} + u(1). \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$\begin{aligned} u(t) = {}& -\frac{1}{\Gamma(\alpha)(1-\eta^{\alpha-1})} \int_0^t \bigl(1-\eta^{\alpha-1}\bigr)(t-s)^{\alpha-1} f\bigl(s,u(s)\bigr)\,ds \\ &+ \frac{1}{\Gamma(\alpha)(1-\eta^{\alpha-1})} \biggl\{ \int_0^1 t^{\alpha-1}(1-s)^{\alpha-1} f\bigl(s,u(s)\bigr)\,ds - \int_0^\eta t^{\alpha-1}(\eta-s)^{\alpha-1} f\bigl(s,u(s)\bigr)\,ds \biggr\} \\ &+ u(1)t^{\alpha-1}. \end{aligned}$$

According to (3.3), it is easy to show that (3.2) holds. The proof is completed. □

Lemma 3.2 *For any* $(t,s) \in [0,1] \times [0,1]$, $G(t,s)$ *is continuous, and* $G(t,s) > 0$ *for any* $(t,s) \in (0,1) \times (0,1)$.

Proof The continuity of $G(t,s)$ for $(t,s) \in [0,1] \times [0,1]$ is obvious.

Let

$$g_1(t,s) = t^{\alpha-1}(1-s)^{\alpha-1} - t^{\alpha-1}(\eta-s)^{\alpha-1} - \left(1-\eta^{\alpha-1}\right)(t-s)^{\alpha-1},$$

we only need to show that $g_1(t,s) > 0$ for $0 \le s \le \min\{t,\eta\} \le 1$, the rest of the proof is similar or obvious. From the definition of $g_1(t,s)$, we have

$$\begin{aligned} g_1(t,s) &= t^{\alpha-1}\left\{(1-s)^{\alpha-1} - (\eta-s)^{\alpha-1} - \left(1-\eta^{\alpha-1}\right)\left(1-\frac{s}{t}\right)^{\alpha-1}\right\} \\ &\ge t^{\alpha-1}\left\{(1-s)^{\alpha-1} - (\eta-s)^{\alpha-1} - \left(1-\eta^{\alpha-1}\right)(1-s)^{\alpha-1}\right\} \\ &\ge t^{\alpha-1}\left\{\eta^{\alpha-1}(1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}\right\} \\ &\ge t^{\alpha-1}\left\{(\eta-\eta s)^{\alpha-1} - (\eta-s)^{\alpha-1}\right\} > 0 \end{aligned}$$

for $0 \le s \le \min\{t,\eta\} \le 1$. The proof is completed. □

Let

$$G(t,s) = t^{\alpha-1}G^*(t,s),$$

then

$$G^*(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha-1} - (\eta-s)^{\alpha-1} - (1-\eta^{\alpha-1})(1-\frac{s}{t})^{\alpha-1}}{\Gamma(\alpha)(1-\eta^{\alpha-1})}, \\ \quad 0 \le s \le \min\{t,\eta\} \le 1; \\ \dfrac{(1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}}{\Gamma(\alpha)(1-\eta^{\alpha-1})}, \quad 0 \le t \le s \le \eta \le 1; \\ \dfrac{(1-s)^{\alpha-1} - (1-\eta^{\alpha-1})(1-\frac{s}{t})^{\alpha-1}}{\Gamma(\alpha)(1-\eta^{\alpha-1})} \quad 0 \le \eta \le s \le t \le 1; \\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\eta^{\alpha-1})}, \quad 0 \le \max\{t,\eta\} \le s \le 1. \end{cases} \tag{3.6}$$

The new Green's function $G^*(t,s)$ has the following properties:

Lemma 3.3 $G^*(t,s)$ *is continuous for* $(t,s) \in (0,1) \times (0,1)$, *and*

$$\lim_{t\to 0} G^*(t,s) := G^*(0,s) = \begin{cases} \dfrac{1}{\Gamma(\alpha)(1-\eta^{\alpha-1})}\left\{(1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}\right\}, \quad 0 \le s \le \eta, \\ \dfrac{1}{\Gamma(\alpha)(1-\eta^{\alpha-1})}(1-s)^{\alpha-1}, \quad \eta \le s \le 1. \end{cases}$$

Furthermore, $G^*(t,s) > 0$ *for* $(t,s) \in (0,1) \times (0,1)$.

Lemma 3.4 *For any* $s \in (0,1)$, $G^*(t,s)$ *is nonincreasing with respect to* $t \in [0,1]$. *Especially, for any* $s \in [0,1]$, $\frac{\partial G^*}{\partial t} \le 0$ *for* $t \in [s,1]$, *and* $\frac{\partial G^*}{\partial t} = 0$ *for* $t \in [0,s]$. *That is* $G^*(1,s) \le G^*(t,s) \le$

$G^*(s,s)$, *where*

$$G^*(1,s) = \frac{1}{\Gamma(\alpha)(1-\eta^{\alpha-1})} \begin{cases} \eta^{\alpha-1}(1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}, & 0 \le s \le \eta, \\ \eta^{\alpha-1}(1-s)^{\alpha-1}, & \eta \le s \le 1, \end{cases} \tag{3.7}$$

and

$$G^*(s,s) = \frac{1}{\Gamma(\alpha)(1-\eta^{\alpha-1})} \begin{cases} (1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}, & 0 \le s \le \eta, \\ (1-s)^{\alpha-1}, & \eta \le s \le 1. \end{cases} \tag{3.8}$$

Let

$$u(t) = t^{\alpha-1}x(t). \tag{3.9}$$

Then, $u(1) = x(1)$, we have from Lemma 3.1, (3.6) and (3.9) that the integral Equation (3.2) can be rewritten as follows:

$$x(t) = \int_0^1 G^*(t,s)f\bigl(s, s^{\alpha-1}x(s)\bigr)\,ds + x(1). \tag{3.10}$$

Let

$$y(t) = x(t) - x(1). \tag{3.11}$$

Then, $y(1) = 0$ and (3.10) is equivalent to the following

$$y(t) = \int_0^1 G^*(t,s)f\bigl(s, s^{\alpha-1}\bigl(y(s) + x(1)\bigr)\bigr)\,ds. \tag{3.12}$$

We can divide our proof into the following two steps:

First, we replace $x(1)$ by any real number μ, then (3.12) can be rewritten as

$$y(t) = \int_0^1 G^*(t,s)f\bigl(s, s^{\alpha-1}\bigl(y(s) + \mu\bigr)\bigr)\,ds. \tag{3.13}$$

It suffices to show that for any given real number μ, (3.13) has a solution $y(t)$, which implies that Equation (1.1) has a solution $u(t)$ which satisfies the first boundary value condition $u(0) = 0$.

Second, we show that there exists a μ such that the solution $y(t)$ of (3.13) satisfies $y(1) = 0$, which implies that the solution $u(t)$ of (1.1) also satisfies the boundary value condition $u(1) = \frac{1}{\eta^{\alpha-1}}u(\eta)$.

In this section, we will prove the first step. For convenience sake, we define an operator T on the set Ω as follows:

$$Ty(t) = \int_0^1 G^*(t,s)f\bigl(s, s^{\alpha-1}\bigl(y(s) + \mu\bigr)\bigr)\,ds. \tag{3.14}$$

Lemma 3.5 *Suppose that* $f \in C([0,1] \times R, R)$*, and (2.4) hold, then the operator* T *is completely continuous in* Ω*.*

Proof It is easy to show that the operator T maps Ω into itself. We divide the proof into the following three steps.

Step 1. $Ty(t)$ is continuous with respect to $y(t) \in \Omega$.

In fact, suppose that $\{y_n(t)\}$ is a sequence in Ω, and $\{y_n(t)\}$ converges to $y(t) \in \Omega$. Since $f(t, t^{\alpha-1}y)$ is continuous with respect to $y \in R$, and it is obvious that $G^*(t,s)$ is uniformly continuous with respect to $(t,s) \in [0,1] \times [0,1]$ from Lemma 3.3, then for any $\varepsilon > 0$, there exists an integer N, when $n > N$,

$$\left\| f\left(t, t^{\alpha-1}\left(y_n(t) + \mu\right)\right) - f\left(t, t^{\alpha-1}\left(y(t) + \mu\right)\right) \right\| \le \frac{\varepsilon}{\int_0^1 G^*(t,s)\,ds}, \tag{3.15}$$

which follows from (3.14)-(3.15) that

$$\begin{aligned} \left\|(Ty_n)(t) - (Ty)(t)\right\| &= \left\| \int_0^1 G^*(t,s)\left\{f\left(s, s^{\alpha-1}\left(y_n(s) + \mu\right)\right) - f\left(s, s^{\alpha-1}\left(y(s) + \mu\right)\right)\right\} ds \right\| \\ &\le \int_0^1 G^*(t,s)\,ds \left\| f\left(t, t^{\alpha-1}\left(y_n(t) + \mu\right)\right) - f\left(t, t^{\alpha-1}\left(y(t) + \mu\right)\right) \right\| \\ &\le \varepsilon. \end{aligned}$$

Thus, the operator T is continuous in Ω.

Step 2. T maps bounded set in Ω into bounded set.

Suppose that $B \in \Omega$ is a bounded set with $\|y(t)\| \le r$ for any $y \in B$. Then, we have from (2.4) and (3.14) that

$$\begin{aligned} \left\|(Ty)(t)\right\| &= \left\| \int_0^1 G^*(t,s) f\left(s, s^{\alpha-1}\left(y(s) + \mu\right)\right) ds \right\| \\ &\le \int_0^1 G^*(t,s)a(s)\,ds + \int_0^1 G^*(t,s)b(s)\left|y(s) + \mu\right|^p ds \\ &\le \int_0^1 G^*(t,s)a(s)\,ds + \int_0^1 G^*(t,s)b(s)\,ds\left(\left\|y(t)\right\| + \|\mu\|\right)^p \\ &\le \int_0^1 G^*(t,s)a(s)\,ds + \int_0^1 G^*(t,s)b(s)\,ds\left(r + \|\mu\|\right)^p := l. \end{aligned}$$

This gives that the operator T maps bounded set into bounded set in Ω.

Step 3. T is equicontinuous in Ω.

It suffices to show that for any $y(t) \in B$ and any $0 < t_1 < t_2 < 1$, $Ty(t_1) \to Ty(t_2)$ as $t_1 \to t_2$. We consider the following three cases:

(i) $0 < t_1 < t_2 < \eta$;
(ii) $0 < t_1 < \eta < t_2$;
(iii) $0 < \eta < t_1 < t_2$.

We only prove the case (i), the rest two cases are similar. Since B is bounded, then there exists a $M > 0$ such that $f \le M$. According to (3.14), we have

$$\begin{aligned} &\left\|(Ty)(t_1) - (Ty)(t_2)\right\| \\ &\quad \le \int_0^1 \left|G^*(t_1,s) - G^*(t_2,s)\right| \left|f\left(s, s^{\alpha-1}\left(y(s) + \mu\right)\right)\right| ds \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{t_1} \frac{1}{\Gamma(\alpha)}\left[\left(1-\frac{s}{t_2}\right)^{\alpha-1} - \left(1-\frac{s}{t_1}\right)^{\alpha-1}\right]\left|f\left(s, s^{\alpha-1}\left(y(s)+\mu\right)\right)\right| ds \\
&\quad + \int_{t_1}^{t_2} \frac{1}{\Gamma(\alpha)}\left(1-\frac{s}{t_2}\right)^{\alpha-1}\left|f\left(s, s^{\alpha-1}\left(y(s)+\mu\right)\right)\right| ds + \int_{t_2}^{1} 0\, ds \\
&\le \frac{M}{\Gamma(\alpha)}\left(\int_0^{t_1}\left[\left(1-\frac{s}{t_2}\right)^{\alpha-1} - \left(1-\frac{s}{t_1}\right)^{\alpha-1}\right] ds + \int_{t_1}^{t_2}\left(1-\frac{s}{t_2}\right)^{\alpha-1} ds\right) \\
&= \frac{M}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha} \to 0 \quad \text{as } t_1 \to t_2.
\end{aligned}
$$

According to Step 1-Step 3, the operator T is completely continuous in Ω. The proof is completed. □

Further, we have

Lemma 3.6 *Suppose that $f \in C([0,1] \times R, R)$, and (2.4) and (2.6) holds, then, for any real number μ, the integral Equation (3.13) has at least one solution.*

Proof We only need to show that the operator T is priori bounded. Let

$$
r = \max\left\{1, \frac{\int_0^1 G^*(s,s)a(s)\,ds + \int_0^1 G^*(s,s)b(s)\,ds|\mu|^p}{1 - \int_0^1 G^*(s,s)b(s)\,ds}\right\}. \tag{3.16}
$$

Define a set $K \in \Omega$ as follows

$$
K = \left\{y \in \Omega \mid \left\|y(t)\right\| \le r\right\}. \tag{3.17}
$$

To show the existence of a fixed point of T by Lemma 2.3, we need to verify that the second possibility in Lemma 2.3 cannot happen.

In fact, assume that there exists $y \in \partial K$ with $\|y(t)\| = r$ and $\gamma \in (0,1)$ such that $y = \gamma Ty$. It follows that

$$
y(t) = \gamma\left|(Ty)(t)\right| = \gamma \int_0^1 G^*(t,s) f\left(s, s^{\alpha-1}\left(y(s)+\mu\right)\right) ds,
$$

and

$$
\begin{aligned}
\left\|y(t)\right\| &= \left\|\gamma \int_0^1 G^*(t,s) f\left(s, s^{\alpha-1}\left(y(s)+\mu\right)\right) ds\right\| \\
&\le \gamma \int_0^1 G^*(s,s)\left|f\left(s, s^{\alpha-1}\left(y(s)+\mu\right)\right)\right| ds \\
&< \int_0^1 G^*(s,s)a(s)\,ds + \int_0^1 G^*(s,s)b(s)\,ds\left\|y(t)+\mu\right\|^p \\
&\le \int_0^1 G^*(s,s)a(s)\,ds + \int_0^1 G^*(s,s)b(s)\,ds\|\mu\|^p + \int_0^1 G^*(s,s)b(s)\,ds\|r\|^p \\
&\le \int_0^1 G^*(s,s)a(s)\,ds + \int_0^1 G^*(s,s)b(s)\,ds\|\mu\|^p + \int_0^1 G^*(s,s)b(s)\,ds\|r\| \\
&\le \|r\|.
\end{aligned} \tag{3.18}
$$

Here we have the use of the inequality

$$(a+b)^p \le a^p + b^p \quad \text{for } a,b \ge 0, 0 \le p \le 1.$$

It is obvious that (3.18) contradicts our assumption that $\|y(t)\| = r$. Therefore, by Lemma 2.3, it follows that T has a fixed point $y \in \overline{K}$. Hence, the integral Equation (3.14) has at least a solution $y(t)$. The proof is completed. □

4 The proof of the main results

Now, we prove Theorem 2.1 by Lemma 3.4-3.5 and the intermediate value theorem.

Proof of Theorem 2.1 It is obvious that the right-hand side of (3.14) is continuously dependent on the parameter μ, so we need to find a μ such that $y(1) = 0$, which implies that $u(1) = \mu$.

For any given real number μ, we rewrite (3.13) as follows:

$$y_\mu(t) = \int_0^1 G^*(t,s) f\big(s, s^{\alpha-1}\big(y_\mu(s) + \mu\big)\big)\, ds, \quad t \in [0,1]. \tag{4.1}$$

From (4.1), it suffices to show that there exists a μ such that

$$L(\mu) := y_\mu(1) = \int_0^1 G^*(1,s) f\big(s, s^{\alpha-1}\big(y_\mu(s) + \mu\big)\big)\, ds. \tag{4.2}$$

It is obvious that $y_\mu(1)$ is continuously dependent on the parameter μ. In order to prove that there exists a μ^* such that $y_{\mu^*}(1) = 0$, we only need to show that $\lim_{\mu\to\infty} L(\mu) = \infty$, and $\lim_{\mu\to-\infty} L(\mu) = -\infty$.

Now, we show that $\lim_{\mu\to\infty} L(\mu) = \infty$. On the contrary, we assume that $\underline{\lim}_{\mu\to\infty} L(\mu) < \infty$. Then, there exists a sequence $\{\mu_n\}$, $\lim_{n\to\infty} \mu_n = \infty$ such that $\lim_{\mu_n\to\infty} L(\mu_n) < \infty$, which implies that the sequence $\{L(\mu_n)\}$ is bounded from above. Notice that the function $f(t, t^{\alpha-1}y)$ is continuous with respect to $t \in [0,1]$ and $y \in R$. We first claim that it is impossible to have

$$f\big(t, t^{\alpha-1}\big(y_{\mu_n}(t) + \mu_n\big)\big) \ge 0 \quad \text{for all } t \in [0,1] \tag{4.3}$$

as μ_n is large enough. Indeed, assume that (4.3) is true. Then, by (4.1), we have

$$y_{\mu_n}(t) \ge 0 \tag{4.4}$$

for all $t \in [0,1]$. Thus, we get

$$\lim_{\mu_n\to\infty} \big(y_{\mu_n}(t) + \mu_n\big) = \infty \tag{4.5}$$

for all $t \in [0,1]$. Since we have assumed in (H) that

$$\lim_{u\to\infty} f\big(t, t^{\alpha-1}u\big) = \infty, \quad t \in (0,1), \tag{4.6}$$

by (4.2), (4.5)-(4.6), we have

$$\begin{aligned}\lim_{\mu_n\to\infty} y_{\mu_n}(1) &= \lim_{\mu_n\to\infty}\int_0^1 G^*(1,s)f\bigl(s,s^{\alpha-1}\bigl(y_{\mu_n}(s)+\mu_n\bigr)\bigr)\,ds\\ &= \lim_{\mu_n\to\infty}\int_{\frac14}^{\frac34} G^*(1,s)f\bigl(s,s^{\alpha-1}\bigl(y_{\mu_n}(s)+\mu_n\bigr)\bigr)\,ds \\ &= \infty,\end{aligned} \tag{4.7}$$

which contradicts our assumption.

Now, for large μ_n, we define

$$I_n = \bigl\{t\in[0,1] | f\bigl(t,t^{\alpha-1}\bigl(y_{\mu_n}(t)+\mu_n\bigr)\bigr) < 0\bigr\}.$$

Then, I_n is not empty.

Further, we divide the set I_n into two sets $\widetilde{I}_n$ and $\widehat{I}_n$ as follows:

$$\widetilde{I}_n = \bigl\{t\in I_n | y_{\mu_n}(t)+\mu_n > 0\bigr\},\qquad \widehat{I}_n = \bigl\{t\in I_n | y_{\mu_n}(t)+\mu_n \le 0\bigr\}.$$

It is easy to know that $\widetilde{I}_n\cap\widehat{I}_n = \phi$, and $\widetilde{I}_n\cup\widehat{I}_n = I_n$, and we have from (H) that $\widehat{I}_n$ is not empty.

From (H) again, the function $f(t,t^{\alpha-1}u)$ is bounded below by a constant for $t\in[0,1]$ and $u\in[0,\infty)$. Thus, there exists a constant M (<0), independent of t and μ_n, such that

$$f\bigl(t,t^{\alpha-1}\bigl(y_{\mu_n}(t)+\mu_n\bigr)\bigr)\ge M,\quad t\in\widetilde{I}_n. \tag{4.8}$$

Let

$$\overline{m}(\mu_n) = \min_{t\in I_n} y_{\mu_n}(t).$$

From the definitions of $\widetilde{I}_n$ and $\widehat{I}_n$, we have

$$\overline{m}(\mu_n) = \min_{t\in\widehat{I}_n} y_{\mu_n}(t) = -\bigl\|y_{\mu_n}(t)\bigr\|_{\widehat{I}_n},$$

and it follows that $\overline{m}(\mu_n)\to-\infty$ as $\mu_n\to\infty$ (since if $\overline{m}(\mu_n)$ is bounded below by a constant as $\mu_n\to\infty$, then (4.7) holds). Therefore, we can choose μ_{n_1} large enough so that

$$\overline{m}(\mu_n) < \min\left\{-1, \frac{M\int_0^1 G^*(s,s)\,ds - \int_0^1 G^*(s,s)a(s)\,ds}{1-\int_0^1 G^*(s,s)b(s)\,ds}\right\} \tag{4.9}$$

for $n > n_1$. From (H), (4.1), (4.8)-(4.9), and the definitions of $\widetilde{I}_n$ and $\widehat{I}_n$, for any $\mu_n > \mu_{n_1}$, we have

$$\begin{aligned} y_{\mu_n}(t) &= \int_0^1 G^*(t,s)f\bigl(s,s^{\alpha-1}\bigl(y_{\mu_n}(s)+\mu_n\bigr)\bigr)\,ds\\ &\ge \int_{I_n} G^*(s,s)f\bigl(s,s^{\alpha-1}\bigl(y_{\mu_n}(s)+\mu_n\bigr)\bigr)\,ds\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{\widetilde{I}_n} G^*(s,s) f\bigl(s, s^{\alpha-1}\bigl(y_{\mu_n}(s)+\mu_n\bigr)\bigr)\,ds \\
&\quad + \int_{\widehat{I}_n} G^*(s,s)\bigl(-a(s)-b(s)\bigl|y_{\mu_n}(s)+\mu_n\bigr|^p\bigr)\,ds \\
&\geq \left(M\int_{\widetilde{I}_n} G^*(s,s)\,ds - \int_{\widehat{I}_n} G^*(s,s)a(s)\,ds\right) \\
&\quad - \int_{\widehat{I}_n} G^*(s,s)b(s)\,ds\bigl\|y_{\mu_n}(t)+\mu_n\bigr\|^p,
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
y_{\mu_n}(t) &\geq M\int_0^1 G^*(s,s)\,ds - \int_0^1 G^*(s,s)a(s)\,ds \\
&\quad - \int_0^1 G^*(s,s)b(s)\,ds\bigl\|y_{\mu_n}(t)\bigr\|_{I_n}^p \\
&\geq M\int_0^1 G^*(s,s)\,ds - \int_0^1 G^*(s,s)a(s)\,ds \\
&\quad + \int_0^1 G^*(s,s)b(s)\,ds\overline{m}(\mu_n), \quad t\in I_n,
\end{aligned}
$$

which implies that

$$
\overline{m}(\mu_n) \geq \frac{M\int_0^1 G^*(s,s)\,ds - \int_0^1 G^*(s,s)a(s)\,ds}{1-\int_0^1 G^*(s,s)b(s)\,ds}.
$$

This contradicts (4.9).

Now, we have proved that $\lim_{\mu\to\infty} L(\mu)=\infty$. By a similar method, we can prove that $\lim_{\mu\to-\infty} L(\mu)=-\infty$. The detail is omitted.

Notice that $L(\mu)$ is continuous with respect to $\mu\in(-\infty,\infty)$. It follows from the intermediate value theorem [11] that there exists a $\mu^*\in(-\infty,\infty)$ such that $L(\mu^*)=0$, that is $y(1)=y_{\mu^*}(1)=0$, which satisfies the second boundary value condition of (1.2). The proof is completed. □

5 Examples

Example 5.1 Consider the following boundary value problem

$$
\begin{cases}
D^{3/2}u(t)+t^2+\dfrac{u}{2}=0, & t\in[0,1], \\
u(0)=0, \qquad u(1)=2^{\frac{1}{2}}u\left(\dfrac{1}{2}\right),
\end{cases}
\tag{5.1}
$$

where

$$
\alpha=3/2, \qquad \eta=\frac{1}{2}, \qquad 2^{\frac{1}{2}}\cdot\left(\frac{1}{2}\right)^{\frac{3}{2}-1}=1,
$$

and

$$
f(t,u)=t^2+\frac{u}{2}, \qquad f\bigl(t,t^{\frac{1}{2}}u\bigr)=t^2+t^{\frac{1}{2}}\frac{u}{2}, \qquad b(t)=\frac{t^{\frac{1}{2}}}{2}.
$$

It is easy to show that

$$\lim_{u\to\pm\infty} f\left(t, t^{\frac{1}{2}}u\right) = \pm\infty, \quad t \in (0,1),$$

and

$$\begin{aligned}\int_0^1 G^*(s,s)b(s)\,ds &\le \frac{1}{2}\cdot\frac{1}{\Gamma(\frac{3}{2})(1-(\frac{1}{2})^{\frac{1}{2}})}\int_0^1 (1-s)^{\frac{1}{2}}s^{\frac{1}{2}}\,ds\\ &= \frac{1}{2}\cdot\frac{1}{\Gamma(\frac{3}{2})(1-(\frac{1}{2})^{\frac{1}{2}})}\cdot\frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(3)}\\ &\approx 0.756 < 1.\end{aligned}$$

Thus, the conditions of Theorem 2.1 are satisfied. Therefore, the problem (5.1) has at least a nontrivial solution.

Competing interests
The authors declare that they have no competing interests.

Author's contributions
Each of the authors, ZO and GL contributed to each part of this study equally and read and approved the final version of the mnanuscript.

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